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Polynomial Interpolation in Several Complex Variables

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1. INTRODUCTION

In this paper we shall prove two interpolation theorems about polynomials in several complex variables. Our results will be applied elsewhere to a problem of Diophantine approximation involving Abelian functions. They are presented here separately on account of their possible independent interest.

For a positive integer n we denote by $\mathbb{C}^n$ the complex n-space equipped with the Euclidean norm $|\mathbf{z}|$ defined for $\mathbf{z} = (z_1, ..., z_n)$ by

$$|\mathbf{z}|^2 = |z_1|^2 + ... + |z_n|^2.$$

Let $P(\mathbf{z}) = P(z_1, ..., z_n)$ be a polynomial in $z_1, ..., z_n$ with complex coefficients. In the first half of this paper we consider the question of determining the general growth of $P(\mathbf{z})$ from its behaviour on a given set $\mathscr{S}$. More precisely, let $\mathfrak{M}(P, \mathscr{S})$ denote the supremum of $|P(\mathbf{z})|$ on a bounded set $\mathscr{S}$, and write $\mathscr{D}^n$ for the unit polydisc defined by the inequalities

$$|z_1| \leqslant 1, ..., |z_n| \leqslant 1.$$

We shall obtain fairly good estimates for $\mathfrak{M}(P, \mathscr{D}^n)$ in terms of $\mathfrak{M}(P, \mathscr{S})$ provided $\mathscr{S}$ satisfies certain conditions. Our main result (Theorem A) is concerned with finite sets $\mathscr{S}$, although to establish this result we shall also have to investigate analogous problems for sets of positive measure.

In Appendix 2 of my thesis [5] I proved the following theorem, in which $\mathscr{B}$ denotes the unit ball defined by $|\mathbf{z}| \leqslant 1$. Let $\mathscr{S}$ be a finite subset of $\mathscr{B}$ containing m points with minimum distance between distinct points at least $\delta \leqslant 1$, and suppose $P(\mathbf{z})$ is of degree at most d in each variable. Then there are positive constants c_1, c_2, depending only on n, such that if

$$m\delta^{2n-2} \geqslant c_1 d \tag{1}$$

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then the absolute values of the coefficients of $P(\mathbf{z})$ do not exceed

$$(c_2 d/\delta)^{nd}\, \mathfrak{M}(P, \mathscr{S}). \tag{2}$$

It is not difficult to deduce a similar bound for $\mathfrak{M}(P, \mathscr{D}^n)$ with, say, $2c_2$ instead of c_2.

Now in applying this result for large d it is impossible to avoid a factor of the order d^d in (2). Theorem A shows that in favourable circumstances we can replace this by a factor of the order $c_3{}^d$ for some c_3 independent of d. Although this is only a slight improvement, it represents a best possible dependence on d; for example, the polynomial $P(z_1, \ldots, z_n) = 2^{nd} z_1{}^d \ldots z_n{}^d$ satisfies $\mathfrak{M}(P, \mathscr{S}) \leqslant 1$ for any finite subset $\mathscr{S}$ of the polydisc $|z_i| \leqslant \frac{1}{2}$ $(1 \leqslant i \leqslant n)$. The exact statement of our result is as follows, in which the separation of a finite set $\mathscr{S}$ is defined (not quite as in [5]) as the minimum distance between distinct points of $\mathscr{S}$.

THEOREM A. *Let $\mathscr{S}$ be a finite subset of $\mathscr{B}$ with cardinality $m > 1$ and separation δ satisfying*

$$m\delta^{2n-2} \geqslant 2^{7n} d, \qquad m\delta^{2n} \geqslant n^n \theta^2$$

for some positive integer d and some positive number θ. Then for any polynomial $P(\mathbf{z})$ of degree at most d in each variable we have

$$\mathfrak{M}(P, \mathscr{D}^n) \leqslant (2^{10n}/\theta)^{nd}\, \mathfrak{M}(P, \mathscr{S}).$$

It follows immediately from Cauchy's integral formula (see Lemma 1 below) that the same inequality holds for the absolute values of the coefficients of $P(\mathbf{z})$. Also by taking the maximum value of θ in this inequality we see that the factor $(c_2 d/\delta)^{nd}$ in (2) can be replaced by $(c_4/d^{1/2}\delta)^{nd}$. Thus if δ is of the same order of magnitude as $d^{-1/2}$ our claims for the improved dependence on d are justified.

The proof of Theorem A will be given in section 4, where we shall also deduce the following corollary.

COROLLARY A. *Let $\mathscr{S}$ be a subset of $\mathscr{B}$ containing a point within $2^{-7n} n^{-n/2} d^{-1/2}$ of each point of $\mathscr{B}$ for some positive integer d. Then for any polynomial $P(\mathbf{z})$ of degree at most d in each variable we have*

$$\mathfrak{M}(P, \mathscr{D}^n) \leqslant (2^{13} n)^{n^2 d}\, \mathfrak{M}(P, \mathscr{S}).$$

This yields, in particular, an explicit form of one of the conjectures on p. 123 of [5], according to which there cannot be a zero of $P(\mathbf{z})$ within $c_5 d^{-1/2}$ of each point of $\mathscr{B}$ unless $P(\mathbf{z})$ is identically zero. The other conjecture, relating to the points of $\mathscr{B}$ with real components, was recently established by Moreau in [7], together with a refinement exactly analogous to our corollary.

In the second half of this paper we apply Theorem A to a special case of the following problem. Let $\mathscr{S}$ be a finite subset of $\mathbb{C}^n$, and let $a(\mathbf{s})$ be complex numbers indexed by points $\mathbf{s}$ of $\mathscr{S}$. We seek the simplest polynomial such that

$$P(\mathbf{s}) = a(\mathbf{s}) \tag{3}$$

for all $\mathbf{s}$. Again let m and δ be the cardinality and separation of $\mathscr{S}$. The very elementary argument of Lemma 19 of [6] (see also Lemma 2 of Appendix 2 of [4]) shows that there exists a polynomial $P(\mathbf{z})$ of degree at most $m - 1$ in each variable satisfying (3). Furthermore, if $\delta \leqslant 1$ and the points $\mathbf{s}$ of $\mathscr{S}$ satisfy $|\mathbf{s}| \leqslant r$ for some $r \geqslant 2$, the coefficients of $P(\mathbf{z})$ can be chosen to have absolute values at most

$$(r/\delta)^{c_6 m} \max |a(\mathbf{s})| \tag{4}$$

for some c_6 depending only on n. It is easy to see that the upper bound on the degree is best possible; for example, if $\mathscr{S}$ lies in the subspace defined by $z_2 = \ldots = z_n = 0$ then the problem essentially involves only a single complex variable. Similarly the estimate (4) cannot in general be substantially improved, at least with regard to the exponent $c_6 m$.

However, if $\mathscr{S}$ is a subset of a certain type of lattice (i.e., a discrete subgroup of rank $2n$) in $\mathbb{C}^n$, we shall see (Theorem B below) that in both estimates the number m can sometimes be replaced by $m^{1/n}$. In fact let K' be a totally real extension of the rational field $\mathbb{Q}$ of degree n, and let K be a totally imaginary quadratic extension of K'. We can find n embeddings $\psi_1, \ldots, \psi_n$ of K into $\mathbb{C}$ which induce distinct embeddings of K' into $\mathbb{C}$. Then as α runs over all integers of K, the points in $\mathbb{C}^n$ of the form

$$L(\alpha) = (\alpha^{\psi_1}, \ldots, \alpha^{\psi_n})$$

define a lattice Λ. Such lattices occur naturally in the theory of complex multiplication of Abelian varieties (cf. [10]). In section 6 we shall prove the following theorem, where for brevity we denote by $r\mathscr{S}$ the set of points of the form $r\mathbf{s}$ for some fixed $r \geqslant 0$ and some $\mathbf{s}$ in a set $\mathscr{S}$.

THEOREM B. *Let Λ be a lattice in $\mathbb{C}^n$ of the type described above. There exists a positive constant C, depending only on Λ, with the following property. Suppose $\mathscr{S}$ is a finite subset of Λ contained in $r\mathscr{B}$ for some $r \geqslant 1$. Then for any complex numbers $a(\mathbf{s})$ indexed by points $\mathbf{s}$ of $\mathscr{S}$, we can find a polynomial $P(\mathbf{z})$, of degree at most Cr^2 in each variable, such that $P(\mathbf{s}) = a(\mathbf{s})$ for all $\mathbf{s}$ and*

$$\mathfrak{M}(P, r\mathscr{D}^n) \leqslant C^{r^2} \max |a(\mathbf{s})| .$$

If $\mathscr{S}$ is as large as possible it contains $m \geqslant c_7 r^{2n}$ points for some positive c_7 independent of r; thus the quantity r^2 occurring above can be of order $m^{1/n}$. It is natural to suppose that a similar improvement on the simple estimates of [6] can be obtained for sets $\mathscr{S}$ which satisfy only a weak distribution condition like that of Theorem A. But at present I cannot find a proof even when $\mathscr{S}$ is a subset of an arbitrary lattice in $\mathbb{C}^n$.

For applications we shall need a generalization of Theorem B involving not only the values of $P(\mathbf{z})$ on $\mathscr{S}$ but also those of its derivatives. Since this will be deduced from Theorem B in section 7, we state it as a corollary. For a nonnegative integral vector $\mathbf{m} = (m_1, \ldots, m_n)$ (i.e., with $m_1, \ldots, m_n$ nonnegative integers) we put

$$D^{\mathbf{m}} = (\partial/\partial z_1)^{m_1} \ldots (\partial/\partial z_n)^{m_n}$$

and

$$|\mathbf{m}| = m_1 + \ldots + m_n, \qquad \mathbf{m}! = m_1! \ldots m_n!.$$

COROLLARY B. *Let Λ be a lattice in $\mathbb{C}^n$ of the type described above. There exists a positive constant C, depending only, on Λ, with the following property. Suppose $\mathscr{S}$ is a finite subset of Λ contained in $r\mathscr{B}$ for some $r \geqslant 1$, and k is a positive integer. Then for any complex numbers $a(\mathbf{s}, \mathbf{m})$ indexed by points $\mathbf{s}$ of $\mathscr{S}$ and nonnegative integral vectors $\mathbf{m}$ with $|\mathbf{m}| < k$ we can find a polynomial $P(\mathbf{z})$, of degree at most Ckr^2 in each variable, such that $D^{\mathbf{m}}P(\mathbf{s}) = a(\mathbf{s}, \mathbf{m})$ for all $\mathbf{s}$, $\mathbf{m}$ and*

$$\mathfrak{M}(P, r'\mathscr{D}^n) \leqslant (Cr'/r)^{Ckr^2} \max |a(\mathbf{s}, \mathbf{m})/\mathbf{m}!|$$

for any $r' \geqslant r$.

Note the more general kind of growth inequality appearing in this result.

2. AUXILIARY RESULTS ON POLYNOMIALS

We collect here various types of elementary estimates for polynomials which will be useful later on. They can be established by induction on the number n of complex variables by means of appropriate arguments with the polynomials $P(a_1, \ldots, a_{n-1}, z)$, $P(z_1, \ldots, z_{n-1}, a)$ for fixed $a_1, \ldots, a_{n-1}, a$. Thus we shall give detailed proofs only for $n = 1$. In this case we denote the disc $\mathscr{D}^1$ simply by $\mathscr{D}$.

LEMMA 1. *The coefficients of a polynomial $P(\mathbf{z})$ do not exceed $\mathfrak{M}(P, \mathscr{D}^n)$ in absolute value.*

Proof. For $n = 1$ let $P(z) = p_d z^d + \ldots + p_0$ for some d; then

$$2\pi i p_r = \int z^{-r-1} P(z)\, dz \quad (0 \leqslant r \leqslant d),$$

where the integral is taken around the unit circle $|z| = 1$ in the anti-clockwise sense. This gives the lemma for $n = 1$, and the general statement follows by induction. We could also have used directly the Cauchy integral formula in $\mathbb{C}^n$.

LEMMA 2. *If $P(\mathbf{z})$ is a polynomial of degree at most d in each variable then for any $r \geqslant 1$ we have*

$$\mathfrak{M}(P, r\mathscr{D}^n) \leqslant r^{nd}\mathfrak{M}(P, \mathscr{D}^n).$$

Proof. For $n = 1$ we consider the reciprocal polynomial $Q(z) = z^d P(z^{-1})$. If $\mathscr{C}$ denotes the boundary $|z| = 1$ of $\mathscr{D}$, then by the maximum modulus principle we have $\mathfrak{M}(P, r\mathscr{D}) = \mathfrak{M}(P, r\mathscr{C})$, and the right-hand side of this is just $r^d\mathfrak{M}(Q, r^{-1}\mathscr{C})$. This number clearly does not exceed $r^d\mathfrak{M}(Q, \mathscr{D}) = r^d\mathfrak{M}(Q, \mathscr{C})$, which in turn is equal to $r^d\mathfrak{M}(P, \mathscr{C})$ and so at most $r^d\mathfrak{M}(P, \mathscr{D})$. The general lemma follows by induction on n. Once again a direct proof is possible using the maximum modulus principle in $\mathbb{C}^n$ (see [5 p. 85]).

LEMMA 3. *If $P(\mathbf{z})$ is a polynomial of degree at most d in each variable which has no zeros in $\mathscr{D}^n$ then*

$$\mathfrak{M}(P, \mathscr{D}^n) \leqslant 2^{3nd} | P(\mathbf{0}) | .$$

Proof. (cf. [5, Lemma A7, p. 129]). Suppose at first that $n = 1$. If $P(z)$ does not vanish on $\mathscr{D}$ then the function $\varphi(z) = (P(z))^{-1}$ is analytic on $\mathscr{D}$. It follows from the maximum modulus principle that for each integer r with $0 \leqslant r \leqslant d$ there is a point a_r with $| a_r | = r/d$ such that $| \varphi(a_r) | \geqslant | \varphi(0) |$. Hence $| P(a_r) | \leqslant | P(0) |$. We now use the Lagrange interpolation formula

$$P(z) = \sum_{r=0}^{d} P(a_r)(z - a_0) \ldots (z - a_d)/(a_r - a_0) \ldots (a_r - a_d), \tag{5}$$

where the terms $z - a_r$, $a_r - a_r$ are omitted in the summand corresponding to r $(0 \leqslant r \leqslant d)$. For any s we have

$$| a_r - a_s | \geqslant | \, | a_r | - | a_s | \, | = | r - s |/d,$$

whence

$$\prod_{s \neq r} | a_r - a_s | \geqslant r!\,(d - r)!\, d^{-d}.$$

Also if $|z| \leqslant 1$ we find that the numerators in (5) satisfy

$$|(z - a_0) \dots (z - a_d)| \leqslant \prod_{r=1}^{d} (1 + r/d) = d^{-d}(2d)!/d!.$$

Hence (5) yields

$$\mathfrak{M}(P, \mathscr{D}) \leqslant |P(0)| \binom{2d}{d} \sum_{r=0}^{d} \binom{d}{r} \leqslant 2^{3d} |P(0)|.$$

This proves Lemma 3 for $n = 1$, and the general assertion follows by induction on n, since for fixed $a_1, \dots, a_{n-1}, a$ in $\mathscr{D}$, the polynomials $P(a_1, \dots, a_{n-1}, z)$, $P(z_1, \dots, z_{n-1}, a)$ do not vanish on $\mathscr{D}$, $\mathscr{D}^{n-1}$ respectively. Note that if $P(\mathbf{z})$ has no zeros in a polydisc $\mathscr{S}$ of radius r centred at $\mathbf{s}$, this result implies that $\mathfrak{M}(P, \mathscr{S}) \leqslant 2^{3nd} |P(\mathbf{s})|$ independently of r.

3. Sets of Positive Measure

Let $\mathscr{S}$ be a subset of $\mathscr{D}^n$ with positive Lebesgue measure. In this section we obtain some estimates for the growth of a polynomial $P(\mathbf{z})$ in terms of $\mathfrak{M}(P, \mathscr{S})$. In the case of a single complex variable such results go back at least to Pólya (see below), and related inequalities for several complex variables occur in work of Bishop [1] (see also [8, p. 133]).

Let μ^n denote the usual Lebesgue measure in $\mathbb{C}^n$, so that

$$\mu^n(\mathscr{D}^n) = \pi^n, \qquad \mu^n(\mathscr{B}) = \pi^n/n!,$$

and write $\mu = \mu^1$. Pólya [9] proved the following theorem. If $P(z)$ is a polynomial of degree d in a single complex variable with leading coefficient unity, then for any $M \geqslant 0$ the set of points z satisfying $|P(z)| \leqslant M$ has measure at most $\pi M^{2/d}$. We shall deduce the next lemma from this result.

Lemma 4. *Let $P(z)$ be a polynomial of degree at most d in a single complex variable and let $\mathscr{S}$ be a subset of $\mathscr{D}$ of positive measure σ. Then*

$$\mathfrak{M}(P, \mathscr{D}) \leqslant 2^{4d} \sigma^{-d/2} \mathfrak{M}(P, \mathscr{S}).$$

Proof. After replacing $\mathscr{S}$ by the subset of $\mathscr{D}$ on which $|P(z)| \leqslant \mathfrak{M}(P, \mathscr{S})$, we may suppose that $\mathscr{S}$ is closed. We assume $P \neq 0$. Let a be any point with $|a| = 2$ and $P(a) \neq 0$, and write

$$Q(z) = P(a + 3z), \; R(z) = z^d Q(z^{-1})/P(a),$$

so that $R(z)$ has exact degree d and leading coefficient unity. Correspondingly let $\mathscr{T}$ be the set of points of the form $\frac{1}{3}(s - a)$ for some s in $\mathscr{S}$, and denote by

$\mathscr{U}$ the set of points of the form t^{-1} for some t in $\mathscr{T}$. Then $\mathfrak{M}(Q, \mathscr{T}) = \mathfrak{M}(P, \mathscr{S})$. Also $|t| \geqslant \frac{1}{3}$ for all t in $\mathscr{T}$, so that

$$\mathfrak{M}(R, \mathscr{U}) \leqslant 3^d \mathfrak{M}(Q, \mathscr{T})/|P(a)|. \tag{6}$$

It follows from Pólya's theorem that $\mu(\mathscr{U}) \leqslant \pi M^{2/d}$ where M is the right-hand side of (6). We proceed to prove that $\mu(\mathscr{U}) \geqslant \mu(\mathscr{T})$.

Since $\mathscr{S}$, and therefore $\mathscr{T}$, is closed, so are the sections $\mathscr{T}(r)$ of $\mathscr{T}$ on which $|z| = r$. If $m(r)$ is the angular measure of $\mathscr{T}(r)$, then $m(r) = 0$ for $r < \frac{1}{3}$ and $r > 1$, and Fubini's theorem for indicator functions (see [11, p. 87]) shows that

$$\mu(\mathscr{T}) = \int_{1/3}^{1} m(r)\, dr.$$

The set $\mathscr{U}$ is also closed, and for $1 \leqslant r \leqslant 3$ the analogous section $\mathscr{U}(r)$ is simply the magnification of $\mathscr{T}(r^{-1})$ by the factor r^2. Thus

$$\mu(\mathscr{U}) = \int_{1}^{3} r^2 m(r^{-1})\, dr.$$

Changing the variable using Proposition 3 [11, p. 104], we find that

$$\mu(\mathscr{U}) = \int_{1/3}^{1} r^{-4} m(r)\, dr \geqslant \int_{1/3}^{1} m(r)\, dr = \mu(\mathscr{T}).$$

Next it is clear that $\mu(\mathscr{T}) = \frac{1}{9}\mu(\mathscr{S})$, and so $\mu(\mathscr{U}) \geqslant \frac{1}{9}\sigma$. Comparison of this with the upper bound for $\mu(\mathscr{U})$ obtained above yields $M \geqslant 3^{-d}\pi^{-d/2}\sigma^{d/2}$, or

$$|P(a)| \leqslant 3^{2d}\pi^{d/2}\sigma^{-d/2}\mathfrak{M}(P, \mathscr{S}).$$

Hence this inequality holds for all a with $|a| = 2$, and Lemma 4 follows on appealing to the maximum modulus principle (and noting that the ancient Egyptian approximation 256/81 for π errs in excess).

Next we generalize this result to several complex variables.

LEMMA 5. *Let $P(\mathbf{z})$ be a polynomial of degree at most d in each variable and let $\mathscr{S}$ be a subset of $\mathscr{D}^n$ of positive measure σ. Then*

$$\mathfrak{M}(P, \mathscr{D}^n) \leqslant 2^{4n^2 d}\sigma^{-nd/2}\mathfrak{M}(P, \mathscr{S}).$$

Proof. As usual the proof is by induction on n, the case $n = 1$ being the previous lemma. Assume the result true with n replaced by $n - 1$ for some $n \geqslant 2$, and let P, d, $\mathscr{S}$, σ be as above. As in the proof of Lemma 4, we can assume that $\mathscr{S}$ is closed. For each z in $\mathscr{D}$, let $m(z)$ be the measure of the set of points $(a_1,..., a_{n-1})$ in $\mathscr{D}^{n-1}$ such that $(a_1,..., a_{n-1}, z)$ lies in $\mathscr{S}$. Then $m(z) \leqslant \pi^{n-1}$ and by Fubini's theorem

$$\sigma = \int_{\mathscr{D}} m(z)\, d\mu.$$

We deduce that the set $\mathscr{T}$ of z in $\mathscr{D}$ for which $m(z) \geqslant \sigma/2\pi$ has measure τ at least $\sigma/2\pi^{n-1}$. For we have $m(z) < \sigma/2\pi$ on the complement $\mathscr{T}'$ of $\mathscr{T}$ in $\mathscr{D}$, and so

$$\sigma = \int_{\mathscr{T}} m(z)\, d\mu + \int_{\mathscr{T}'} m(z)\, d\mu \leqslant \pi^{n-1}\tau + (\sigma/2\pi)\,\pi.$$

Hence for any t in $\mathscr{T}$ the polynomial $Q(z_1,..., z_{n-1}) = P(z_1,..., z_{n-1}, t)$ satisfies

$$| Q(z_1,..., z_{n-1}) | \leqslant \mathfrak{M}(P, \mathscr{S})$$

on a set in $\mathscr{D}^{n-1}$ of measure at least $\sigma/2\pi$. By our induction hypothesis

$$\mathfrak{M}(Q, \mathscr{D}^{n-1}) \leqslant 2^{4(n-1)^2 d}\,(\sigma/2\pi)^{-(n-1)d/2}\,\mathfrak{M}(P, \mathscr{S}).$$

In other words, for any fixed $(a_1,..., a_{n-1})$ in $\mathscr{D}^{n-1}$ the polynomial $R(z) = P(a_1,..., a_{n-1}, z)$ satisfies

$$\mathfrak{M}(R, \mathscr{T}) \leqslant 2^{4(n-1)^2 d}\,(\sigma/2\pi)^{-(n-1)d/2}\,\mathfrak{M}(P, \mathscr{S}).$$

We deduce from Lemma 4 that

$$\mathfrak{M}(R, \mathscr{D}) \leqslant 2^{4d}(\sigma/2\pi^{n-1})^{-d/2}\,\mathfrak{M}(R, \mathscr{T}) \leqslant 2^{4n^2 d}\sigma^{-nd/2}\mathfrak{M}(P, \mathscr{S}).$$

Thus the same upper bound holds for $\mathfrak{M}(P, \mathscr{D}^n)$, and this completes the proof of Lemma 5.

By slightly more elaborate arguments the estimate of this lemma can be improved with respect to its dependence on both n and σ, and indeed best possible results can be obtained (see [12]). We do not go into this now, however, because our applications involve essentially constant values of these parameters.

4. Proof of Theorem A and Corollary A

Let $P(\mathbf{z})$ be a polynomial of total degree D and consider the divisor in $\mathbb{C}^n$ defined by $P(\mathbf{z}) = 0$. We can construct a $(2n - 2)$-dimensional Hausdorff measure on this divisor which takes multiplicities into account; for $\mathbf{a}$ in $\mathbb{C}^n$ and $r \geqslant 0$ let us write the corresponding measure in the ball $| \mathbf{z} - \mathbf{a} | \leqslant r$ in the form

$$\pi^{n-1} r^{2n-2}\Theta(\mathbf{a}, r)/(n-1)!$$

for some $\Theta(\mathbf{a}, r)$. Then it is known that the function $\Theta(\mathbf{a}, r)$ has the following properties;

(i) $\Theta(\mathbf{a}, r)$ is monotone nondecreasing in r

(ii) $\Theta(\mathbf{a}) = \lim_{r\to 0} \Theta(\mathbf{a}, r)$ is the order of the zero of $P(\mathbf{z})$ at $\mathbf{z} = \mathbf{a}$

(iii) $\lim_{r\to\infty} \Theta(\mathbf{a}, r) = D$ independently of $\mathbf{a}$.

For references see Bombieri and Lang [2].

We now prove Theorem A. Let $\mathscr{S}$ be a finite subset of $\mathscr{B}$ consisting of $m > 1$ points $\mathbf{s}_1, \ldots, \mathbf{s}_m$ with separation $\delta \leqslant 2$ satisfying

$$m\delta^{2n-2} \geqslant 2^{7n} d, \qquad m\delta^{2n} \geqslant n^n \theta^2$$

for some integer $d \geqslant 1$ and some real number $\theta > 0$. Furthermore let $P(\mathbf{z})$ be a polynomial of degree at most d in each variable. Consider the balls $\mathscr{B}_i$ defined by $|\mathbf{z} - \mathbf{s}_i| \leqslant \frac{1}{5}\delta$ $(1 \leqslant i \leqslant m)$, and suppose exactly $l \leqslant m$ of these contain a zero of $P(\mathbf{z})$, without loss of generality those with $1 \leqslant i \leqslant l$. If $\mathbf{t}_i$ is a zero of $P(\mathbf{z})$ in $\mathscr{B}_i$ $(1 \leqslant i \leqslant l)$, then the balls $|\mathbf{z} - \mathbf{t}_i| \leqslant \frac{1}{5}\delta$ are disjoint and contained in $2\mathscr{B}$. We proceed to estimate $\Theta(\mathbf{0}, 2)$ in two ways. On the one hand, by (i) and (iii) we have, since $D \leqslant nd$,

$$\Theta(\mathbf{0}, 2) \leqslant nd.$$

On the other hand, from the measure-theoretic definition of the Θ-function we have

$$2^{2n-2}\Theta(\mathbf{0}, 2) \geqslant (\tfrac{1}{5}\delta)^{2n-2} \sum_{i=1}^{l} \Theta(\mathbf{t}_i, \tfrac{1}{5}\delta),$$

and using (i) and (ii) we see that this is at least

$$(\tfrac{1}{5}\delta)^{2n-2} \sum_{i=1}^{l} \Theta(\mathbf{t}_i) \geqslant l(\tfrac{1}{5}\delta)^{2n-2}.$$

We conclude that

$$l \leqslant (10/\delta)^{2n-2}\, nd < \tfrac{1}{2}m.$$

This means that exactly $m - l > \frac{1}{2}m$ of the balls $\mathscr{B}_i$ do not contain a zero of $P(\mathbf{z})$. Now the polydisc $\mathscr{D}_i$ of radius $\delta/5n^{1/2}$ centred at $\mathbf{s}_i$ lies completely in $\mathscr{B}_i$, whence for each $i > l$ the polynomial $P(\mathbf{z})$ has no zeros in $\mathscr{D}_i$, and so Lemma 3 implies that

$$\mathfrak{M}(P, \mathscr{D}_i) \leqslant 2^{3nd}\, |P(\mathbf{s}_i)| \leqslant 2^{3nd}\mathfrak{M}(P, \mathscr{S}) \qquad (l < i \leqslant m).$$

But the total measure of the sets $\mathscr{D}_i$ $(l < i \leqslant m)$ is

$$(m - l)\,\pi^n(\delta/5n^{1/2})^{2n} \geqslant \tfrac{1}{2}5^{-2n}\pi^n\theta^2,$$

and they all lie in $2\mathscr{B}$. Hence the polynomial $Q(\mathbf{z}) = P(2\mathbf{z})$ satisfies

$$\mathfrak{M}(Q, \mathscr{T}) \leqslant 2^{3nd}\mathfrak{M}(P, \mathscr{S})$$

for a subset $\mathscr{T}$ of $\mathscr{D}^n$ of measure at least $\frac{1}{2}10^{-2n}\pi^n\theta^2$. Applying Lemma 5, we deduce that

$$\mathfrak{M}(P, 2\mathscr{D}^n) = \mathfrak{M}(Q, \mathscr{D}^n) \leqslant 2^{4n^2d}(\tfrac{1}{2}10^{-2n}\pi^n\theta^2)^{-nd/2}2^{3nd}\mathfrak{M}(P, \mathscr{S}).$$

Hence

$$\mathfrak{M}(P, \mathscr{D}^n) \leqslant \mathfrak{M}(P, 2\mathscr{D}^n) \leqslant (2^{10n}/\theta)^{nd}\,\mathfrak{M}(P, \mathscr{S})$$

(on noting this time that the Roman approximation $3\frac{1}{8}$ for π errs in defect). This completes the proof of Theorem A.

To deduce Corollary A we follow [5 p. 127]. Suppose $\mathscr{S}$ is a subset of $\mathscr{B}$ containing a point within $\delta \leqslant 2^{-7n}n^{-n/2}d^{-1/2}$ of each point of $\mathscr{B}$, and let $P(\mathbf{z})$ be a polynomial of degree at most d in each variable. Select an integer k satisfying

$$(4\delta)^{-1} \leqslant k \leqslant (3\delta)^{-1},$$

and consider the points

$$\mathbf{a} = ((\mu_1 + i\nu_1)/k, \ldots, (\mu_n + i\nu_n)/k)$$

as $\mu_1, \nu_1, \ldots, \mu_n, \nu_n$ range over all nonnegative integers not exceeding $k/2n^{1/2}$. There are

$$m \geqslant (k/2n^{1/2})^{2n} \geqslant 2^{-6n}n^{-n}\delta^{-2n}$$

such points, and they all lie in $2^{-1/2}\mathscr{B}$. For each $\mathbf{a}$ let $\mathbf{s}(\mathbf{a})$ be a point of $\mathscr{S}$ nearest $\mathbf{a}$. Since $k^{-1} - 2\delta \geqslant \delta$, the set $\mathscr{S}'$ of points $\mathbf{s}(\mathbf{a})$ has cardinality m and separation at least δ, and it is clearly contained in $\mathscr{B}$. Furthermore we have

$$m\delta^{2n-2} \geqslant 2^{-6n}n^{-n}\delta^{-2} \geqslant 2^{7n}d, \qquad m\delta^{2n} \geqslant n^n\theta^2$$

with $\theta = 2^{-3n}n^{-n}$. Hence we may apply Theorem A to the polynomial $P(\mathbf{z})$ on the set $\mathscr{S}'$, and we conclude that

$$\mathfrak{M}(P, \mathscr{D}^n) \leqslant 2^{13n^2d}n^{n^2d}\mathfrak{M}(P, \mathscr{S}') \leqslant (2^{13}n)^{n^2d}\,\mathfrak{M}(P, \mathscr{S})$$

as required.

5. Lemmas on Algebraic Numbers

We prove Theorem B in the next section. We shall need some elementary facts about algebraic numbers which it is convenient to record separately in this section.

Let K be a totally imaginary quadratic extension of a totally real field K', with

$$[K : \mathbb{Q}] = 2[K' : \mathbb{Q}] = 2n,$$

and choose embeddings $\psi_1, \ldots, \psi_n$ of K into $\mathbb{C}$ that induce distinct embeddings of K' into $\mathbb{C}$. Thus the conjugates of any α in K are given by $\alpha^{\psi_1}, \ldots, \alpha^{\psi_n}$ and their complex conjugates. Our first lemma deals with 'arithmetic progressions' in the ring I of integers of K, that is, congruence classes modulo a fixed element of I.

Lemma 6. *Let π be a prime element of K, and let $\beta_1, \ldots, \beta_l$ be representatives of the nonzero congruence classes of I modulo π. If $\mathfrak{A}$ denotes one of these congruence classes then the sets $\beta_1^{-1}\mathfrak{A}, \ldots, \beta_l^{-1}\mathfrak{A}$ between them contain all elements of I not divisible by π.*

Proof. Suppose $\mathfrak{A}$ consists of all elements of I congruent to α modulo π, so that α is not divisible by π, and let γ be any element of I not divisible by π. Since the nonzero congruence classes of I form a multiplicative group, there exists β_i with $\beta_i\gamma$ congruent to α modulo π, whence γ lies in $\beta_i^{-1}\mathfrak{A}$.

Lemma 7. *For any $\Delta > 0$ there exists a prime element of K all of whose conjugates exceed Δ in absolute value.*

Proof. For a nonzero element α in I let

$$D(\alpha) = (\log|\alpha^{\psi_1}|, \ldots, \log|\alpha^{\psi_n}|)$$

be a point of the real space $\mathbb{R}^n$. Because K has no real embeddings, this gives rise to the well-known Dirichlet map associated with K. The image of the group of units of I is a lattice in the subspace of $\mathbb{R}^n$ consisting of all $(x_1, \ldots, x_n)$ with $x_1 + \ldots + x_n = 0$. It follows by simple geometry that if η is a unit with $D(\eta)$ nearest to $D(\alpha)$ we have for all i, j

$$|\log|\pi^{\psi_i}| - \log|\pi^{\psi_j}|| \leqslant c \tag{7}$$

with $\pi = \alpha\eta^{-1}$ and some constant c depending only on K.

Now there are infinitely many principal prime ideals $\mathfrak{p}$ in K (see [3, p. 214]), and so we can select one with norm at least $e^{2cn}\Delta^{2n}$. Let α be a generator of

$\mathfrak{p}$ and let η be a unit with $D(\eta)$ nearest to $D(\alpha)$. Then $\pi = \alpha\eta^{-1}$ is a prime element of K and we deduce from (7) that for any i

$$\text{Norm}\,\pi = |\pi^{\psi_1} \dots \pi^{\psi_n}|^2 \leqslant e^{2cn} |\pi^{\psi_i}|^{2n}.$$

Since this norm is no smaller than $e^{2cn}\Delta^{2n}$, we find that $|\pi^{\psi_i}| \geqslant \Delta$ and this establishes Lemma 7.

6. Proof of Theorem B

With the notation of the preceding section, we associate to each α in K the complex vector

$$L(\alpha) = (\alpha^{\psi_1}, \dots, \alpha^{\psi_n}).$$

The image $\Lambda = L(I)$ of I is then a lattice in $\mathbb{C}^n$, because it is discrete and of rank $2n$; in fact for nonzero α in I we have

$$|L(\alpha)|^2 = |\alpha^{\psi_1}|^2 + \dots + |\alpha^{\psi_n}|^2 \geqslant n |\alpha^{\psi_1} \dots \alpha^{\psi_n}|^{2/n} = n(\text{Norm}\,\alpha)^{1/n} \geqslant 1.$$

For $r \geqslant 0$ denote by $\Lambda(r)$ the subset of Λ lying in the ball $r\mathscr{B}$; that is, the set of points $\boldsymbol{\lambda}$ in Λ with $|\boldsymbol{\lambda}| \leqslant r$. Thus $\Lambda(r)$ is the origin if $r < 1$. The following lemma contains the most important part of the proof of Theorem B.

Lemma 8. *There exists a positive constant c, depending only on Λ, with the following property. For any $r \geqslant 1$ there is a polynomial $P(\mathbf{z})$, of degree at most cr^2 in each variable, which vanishes at all nonzero points of $\Lambda(r)$ but satisfies*

$$P(\mathbf{0}) = 1, \qquad \mathfrak{M}(P, r\mathscr{D}^n) \leqslant c^{r^2}.$$

Proof. We shall denote by $c_1, \dots$ positive constants depending only on Λ. Let π be a prime element of K, to be specified later, and let a be the minimum of the absolute values of its conjugates. Select representatives $\beta_1, \dots, \beta_l$ of the nonzero congruence classes of I modulo π, and let b be the maximum of the absolute values of all their conjugates.

For brevity we shall say that a point $\boldsymbol{\lambda}$ of Λ is divisible by π if $\boldsymbol{\lambda} = L(\alpha)$ for some α divisible by π. For any $r \geqslant 1$ consider the set $\mathscr{S}$ of points of $\Lambda(2br)$ divisible by π. Since $|L(\alpha)| \leqslant 2br$ implies

$$|L(\pi^{-1}\alpha)| \leqslant a^{-1} |L(\alpha)| \leqslant 2a^{-1}br$$

we see that $\mathscr{S}$ contains at most $c_1(a^{-1}br)^{2n}$ points. Hence if $d \leqslant c_2(a^{-1}br)^2$ is the greatest integer not exceeding $c_1^{1/n}(a^{-1}br)^2$, we can choose the $(d+1)^n > c_1(a^{-1}br)^{2n}$ coefficients of a polynomial $Q(\mathbf{z})$ of degree at most d in each variable such that $Q(\mathbf{z})$ vanishes on $\mathscr{S}$ but is not identically zero.

We now try to apply Theorem A to the polynomial $Q(br\mathbf{z})$ on the subset $(br)^{-1}\Lambda(br)$ of $\mathscr{B}$. Clearly this set contains $m \geqslant c_3(br)^{2n}$ points with separation $\delta \geqslant c_4(br)^{-1}$. It follows that if

$$a \geqslant c_5 = 2^{7n/2}c_2^{1/2}c_3^{-1/2}c_4^{-(n-1)}$$

then indeed Theorem A is applicable in these circumstances. In view of this, we use Lemma 7 to fix π as a prime element of smallest height such that $a \geqslant c_5$ and $a > 1$. We deduce that

$$\mathfrak{M}(Q, br\mathscr{D}^n) \leqslant c_6^{r^2}\mathfrak{M}(Q, \Lambda(br)).$$

Since $Q(\mathbf{z})$ now has degree at most c_7r^2 in each variable we obtain at once using Lemma 2

$$\mathfrak{M}(Q, 2br\mathscr{D}^n) \leqslant 2^{nd}\mathfrak{M}(Q, br\mathscr{D}^n) \leqslant c_8^{r^2}\mathfrak{M}(Q, \Lambda(br)).$$

In other words, there exists a point $-\boldsymbol{\lambda}_0$ in $\Lambda(br)$ such that

$$|Q(-\boldsymbol{\lambda}_0)| \geqslant c_8^{-r^2}\mathfrak{M}(Q, 2br\mathscr{D}^n);$$

in particular, $Q(-\boldsymbol{\lambda}_0) \neq 0$ so that $\boldsymbol{\lambda}_0$ is not divisible by π.

It follows that the polynomial

$$R(\mathbf{z}) = Q(\mathbf{z} - \boldsymbol{\lambda}_0)/Q(-\boldsymbol{\lambda}_0)$$

satisfies

$$\mathfrak{M}(R, br\mathscr{D}^n) \leqslant c_8^{r^2}$$

and has the same degree as $Q(\mathbf{z})$. Furthermore we have $R(\mathbf{0}) = 1$ and by construction $R(\mathbf{z})$ vanishes on the set of points of the form $\boldsymbol{\lambda}_0 + \boldsymbol{\lambda}$ for some $\boldsymbol{\lambda}$ in $\Lambda(2br)$ divisible by π. Now write $\boldsymbol{\lambda}_0 = L(\alpha_0)$ and let $\mathfrak{A}$ be the arithmetic progression consisting of all elements of I congruent to α_0 modulo π. Since $|\boldsymbol{\lambda}_0| \leqslant br$, we find that $R(\mathbf{z})$ vanishes at all points in $br\mathscr{B}$ of the set $L(\mathfrak{A})$.

Next we put

$$S(z_1, \ldots, z_n) = \prod_{i=1}^{l} R(\beta_i^{\psi_1}z_1, \ldots, \beta_i^{\psi_n}z_n)$$

and deduce from the properties of $R(\mathbf{z})$ the following properties of $S(\mathbf{z})$. It has degree at most c_9r^2 in each variable, and, because $(\beta_i^{\psi_1}z_1, \ldots, \beta_i^{\psi_n}z_n)$ lies in $br\mathscr{D}^n$ whenever $(z_1, \ldots, z_n)$ lies in $r\mathscr{D}^n$, we have

$$\mathfrak{M}(S, r\mathscr{D}^n) \leqslant (\mathfrak{M}(R, br\mathscr{D}^n))^l \leqslant c_{10}^{r^2}.$$

Also $S(\mathbf{0}) = 1$ and for each i the polynomial $S(\mathbf{z})$ vanishes at all points $L(\gamma)$ of $L(\beta_i^{-1}\mathfrak{A})$ with $(\beta_i^{\psi_1}\gamma^{\psi_1}, \ldots, \beta_i^{\psi_n}\gamma^{\psi_n})$ in $br\mathscr{B}$ $(1 \leqslant i \leqslant l)$. In particular it vanishes at all points in $r\mathscr{B}$ of the sets $L(\beta_1^{-1}\mathfrak{A}),\ldots, L(\beta_l^{-1}\mathfrak{A})$. Hence from Lemma 6 the polynomial $S(\mathbf{z})$ vanishes on all points of $\Lambda(r)$ not divisible by π.

Finally we extend the range of zeros to all nonzero points of $\Lambda(r)$. Remembering that the foregoing arguments depend on the parameter r, we rename the polynomial $S(\mathbf{z})$ as $S(\mathbf{z}; r)$. Since $a > 1$, there exists a greatest integer K with $a^K \leqslant r$, and for each nonnegative integer $k \leqslant K$ we put

$$S_k(\mathbf{z}) = S((\pi^{\psi_1})^{-k} z_1 ,\ldots, (\pi^{\psi_n})^{-k} z_n; a^{-k}r).$$

We proceed to verify that the polynomial

$$P(\mathbf{z}) = S_0(\mathbf{z}) \ldots S_K(\mathbf{z})$$

satisfies the conditions of Lemma 8. Its degree in each variable does not exceed

$$c_9(r^2 + a^{-2}r^2 + \ldots + a^{-2K}r^2) < c_9r^2/(1 - a^{-2}) = c_{11}r^2.$$

Furthermore, if $(z_1 ,\ldots, z_n)$ lies in $r\mathscr{D}^n$ then $((\pi^{\psi_1})^{-k} z_1 ,\ldots, (\pi^{\psi_n})^{-k} z_n)$ lies in $a^{-k}r\mathscr{D}^n$, so that

$$\mathfrak{M}(S_k , r\mathscr{D}^n) \leqslant c_{10}^{a^{-2k}r^2}.$$

Thus a similar calculation yields

$$\mathfrak{M}(P, r\mathscr{D}^n) \leqslant c_{12}^{r^2}.$$

Also $P(\mathbf{0}) = 1$. To verify the assertion about the zeros of $P(\mathbf{z})$ we note that any nonzero $\boldsymbol{\lambda}$ in $\Lambda(r)$ can be written as $L(\pi^k\alpha)$ for some α in I not divisible by π and some nonnegative integer k. Since

$$1 \leqslant | L(\alpha) | \leqslant a^{-k}r,$$

we must have $k \leqslant K$ and consequently $S(\mathbf{z}; a^{-k}r)$ vanishes at $L(\alpha)$. Hence $S_k(\mathbf{z})$ vanishes at $\boldsymbol{\lambda} = L(\pi^k\alpha)$ and we conclude that $P(\mathbf{z})$ also vanishes at $\boldsymbol{\lambda}$. This completes the proof of Lemma 8.

The proof of Theorem B is now immediate. Suppose $\mathscr{S}$ is a subset of $\Lambda(r)$ for some $r \geqslant 1$, and the $a(\mathbf{s})$ are complex numbers indexed by points $\mathbf{s}$ of $\mathscr{S}$. We use Lemma 8 to construct a polynomial $Q(\mathbf{z})$, of degree at most $4cr^2$ in each variable, which vanishes at all nonzero points of $\Lambda(2r)$ but satisfies

$$Q(\mathbf{0}) = 1, \qquad \mathfrak{M}(Q, 2r\mathscr{D}^n) \leqslant c^{4r^2}.$$

Then clearly the sum

$$P(\mathbf{z}) = \sum a(\mathbf{s})\, Q(\mathbf{z} - \mathbf{s}),$$

taken over all $\mathbf{s}$ in $\mathscr{S}$, fulfills the conditions of Theorem B.

7. Proof of Corollary B

In this section we shall deduce Corollary B from the following lemma.

Lemma 9. *There exists a positive constant c, depending only on Λ, with the following property. For a positive integer k let $a(\mathbf{m})$ be complex numbers indexed by non-negative integral vectors $\mathbf{m}$ with $|\mathbf{m}| < k$. Then for any $r \geq 1$ there is a polynomial $P(\mathbf{z})$, of degree at most ckr^2 in each variable, which has a zero of order at least k at all nonzero points of $\Lambda(r)$ but satisfies $D^{\mathbf{m}}P(\mathbf{0}) = a(\mathbf{m})$ ($|\mathbf{m}| < k$) and*

$$\mathfrak{M}(P, r\mathscr{D}^n) \leq c^{kr^2} \max |a(\mathbf{m})/\mathbf{m}!| .$$

Proof. If $\mathbf{m} = (m_1, ..., m_n)$ let $z^{\mathbf{m}} = z_1^{m_1} \ldots z_n^{m_n}$ and form the sum

$$A(\mathbf{z}) = \sum a(\mathbf{m})\, z^{\mathbf{m}}/\mathbf{m}!$$

taken over all nonnegative integral vectors $\mathbf{m}$ with $|\mathbf{m}| < k$. We use Lemma 8 to construct a polynomial $Q(\mathbf{z})$ with $Q(\mathbf{0}) = 1$ which vanishes at all nonzero points of $\Lambda(r)$. Then the rational function $(Q(\mathbf{z}))^{-k} A(\mathbf{z})$ has a Taylor expansion about the origin. If $R(\mathbf{z})$ denotes the sum of the terms of total degree less than k in this expansion, we claim that the polynomial $P(\mathbf{z}) = (Q(\mathbf{z}))^k R(\mathbf{z})$ satisfies the conditions of the present lemma. It clearly has a zero of order at least k at all nonzero points of $\Lambda(r)$. Also, since

$$(Q(\mathbf{z}))^{-k} A(\mathbf{z}) = R(\mathbf{z}) + w(\mathbf{z})$$

for some power series $w(\mathbf{z})$ with a zero of order at least k at the origin, we have

$$P(\mathbf{z}) = A(\mathbf{z}) - (Q(\mathbf{z}))^k w(\mathbf{z}),$$

and so $D^{\mathbf{m}}P(\mathbf{0}) = D^{\mathbf{m}}A(\mathbf{0}) = a(\mathbf{m})$ whenever $|\mathbf{m}| < k$. We proceed to estimate $\mathfrak{M}(P, r\mathscr{D}^n)$ by means of majorization techniques.

For two formal power series

$$g(\mathbf{z}) = \sum p(\mathbf{m})\, z^{\mathbf{m}}, \qquad h(\mathbf{z}) = \sum q(\mathbf{m})\, z^{\mathbf{m}}$$

with $q(\mathbf{m})$ real, we write $g(\mathbf{z}) \ll h(\mathbf{z})$ if $|p(\mathbf{m})| \leq q(\mathbf{m})$ for all nonnegative integral vectors $\mathbf{m}$. If $h(\mathbf{z})$ converges on $r\mathscr{D}^n$ these inequalities plainly imply that $|g(\mathbf{z})| \leq h(r, ..., r)$ on $r\mathscr{D}^n$.

Now if $A = \max |a(\mathbf{m})/\mathbf{m}!|$ we have

$$A(r\mathbf{z}) \ll Af_k(r\mathbf{z})$$

where $f_k(\mathbf{z})$ is the sum $\sum \mathbf{z}^{\mathbf{m}}$ taken over all non-negative integral vectors $\mathbf{m}$

with $|\mathbf{m}| < k$. Further, from Lemma 1 the coefficients of $Q(r\mathbf{z})$ do not exceed

$$M = \max(1, \mathfrak{M}(Q, r\mathscr{D}^n))$$

in absolute value. This gives

$$Q(r\mathbf{z}) - 1 \ll M(f(\mathbf{z}) - 1)$$

where $f(\mathbf{z})$ is the sum $\sum \mathbf{z}^{\mathbf{m}}$ taken over all nonnegative integral vectors $\mathbf{m}$; that is,

$$f(\mathbf{z}) = (1 - z_1)^{-1} \dots (1 - z_n)^{-1}.$$

It follows that

$$(Q(r\mathbf{z}))^{-k} \ll (1 - M(f(\mathbf{z}) - 1))^{-k} \ll \sum_{j=0}^{\infty} \binom{k+j-1}{j} M^j (f(\mathbf{z}) - 1)^j.$$

We immediately obtain a majorizing series for $(Q(r\mathbf{z}))^{-k} A(r\mathbf{z})$, and by truncating we find that

$$R(r\mathbf{z}) \ll A f_k(r\mathbf{z}) \sum_{j=0}^{k-1} \binom{k+j-1}{j} M^j (f(\mathbf{z}) - 1)^j.$$

On specializing to points of $\frac{1}{2}\mathscr{D}^n$ we get the estimate

$$\mathfrak{M}(R, \tfrac{1}{2}r\mathscr{D}^n) \leqslant A f_k(\tfrac{1}{2}r, \dots, \tfrac{1}{2}r) \sum_{j=0}^{k-1} \binom{k+j-1}{j} M^j (2^n - 1)^j.$$

Now

$$f_k(\tfrac{1}{2}r, \dots, \tfrac{1}{2}r) \leqslant (1 + \tfrac{1}{2}r)^{nk} \leqslant 2^{nkr},$$

and the sum over j does not exceed

$$2^{2k} M^k \sum_{j=0}^{k} \binom{k}{j} (2^n - 1)^j \leqslant 2^{3nk} M^k;$$

thus from Lemma 2 we deduce that

$$\mathfrak{M}(R, r\mathscr{D}^n) \leqslant 2^{nk} \mathfrak{M}(R, \tfrac{1}{2}r\mathscr{D}^n) \leqslant 2^{4nk} 2^{nkr} A M^k \leqslant 2^{5nkr} A M^k.$$

We complete the proof of Lemma 9 by using the estimate of Lemma 8 for the number $\mathfrak{M}(Q, r\mathscr{D}^n)$ in the definition of M.

Finally Corollary B follows from Lemma 9 just as Theorem B follows from Lemma 8. Let $r \geqslant 1, k \geqslant 1$, and let $a(\mathbf{s}, \mathbf{m})$ be complex numbers indexed by elements $\mathbf{s}$ of a subset $\mathscr{S}$ of $\Lambda(r)$ and non-negative integral vectors $\mathbf{m}$ with $|\mathbf{m}| < k$. We can then construct for each $\mathbf{s}$ in $\mathscr{S}$ a polynomial $P_{\mathbf{s}}(\mathbf{z})$ satisfying $D^{\mathbf{m}} P_{\mathbf{s}}(\mathbf{0}) = a(\mathbf{s}, \mathbf{m})$ with zeros of order at least k at all nonzero points of $\Lambda(2r)$, and we take $P(\mathbf{z}) = \sum P_{\mathbf{s}}(\mathbf{z} - \mathbf{s})$. This gives $D^{\mathbf{m}}P(\mathbf{s}) = a(\mathbf{s}, \mathbf{m})$ for all $\mathbf{s}$, $\mathbf{m}$, and

$$\mathfrak{M}(P, r\mathscr{D}^n) \leqslant c^{kr^2} \max |a(\mathbf{s}, \mathbf{m})/\mathbf{m}!| .$$

Since the degree of $P(\mathbf{z})$ is at most ckr^2, the more general estimate

$$\mathfrak{M}(P, r'\mathscr{D}^n) \leqslant (Cr'/r)^{Ckr^2} \max | a(\mathbf{s}, \mathbf{m})/\mathbf{m}! |$$

for $r' \geqslant r$ can be obtained by applying Lemma 2 to the polynomial $P(r\mathbf{z})$ on the polydisc $(r'/r)\, \mathscr{D}^n$. This concludes the proof of Corollary B.

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